Translatable radii of an operator in the direction of another operator II

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Abstract

One of the couple of translatable radii of an operator in the direction of another operator introduced in earlier work[13] is studied in details. A necessary and sufficient condition for a unit vector f to be a stationary vector of the generalized eigenvalue problem $Tf = \lambda Af$ is obtained. Finally a theorem of Williams[16] is generalized to obtain a translatable radius of an operator in the direction of another operator.

1 Introduction.

Let T and A be two bounded linear operators on a complex Hilbert space H with inner product (,) and norm $\|\|$. Consider the generalized eigenvalue problem Tf = λ Af where f $\in$ H and $\lambda \in$ C, λ is called the eigenvalue of the above equation and f the corresponding eigenvector. The non-negative functional

$$M_T(f) = \|Tf - \frac{(Tf, Af)}{(Af, Af)}Af\|, \ provided \ \|Af\| \neq 0,$$

gives the deviation of a unit vector f from being an eigenvector and

$$M_T(A) = sup_{\|f\|=1} \ \{\|Tf - \frac{(Tf, Af)}{(Af, Af)}Af\|\}, \ provided \ 0 \notin \sigma_{app}A,$$

gives the supremum of all those deviations, where $\sigma_{app}A$ is the set of approximate eigenvalues of A.

Geometrically $Tf - \frac{(Tf,Af)}{(Af,Af)}Af$ is the component of Tf perpendicular to Af. For A = I problems related to the concepts considered here have been studied by Bjorck and Thomee[2], Garske[8], Prasanna[14], Fujii and Prasanna[6], Furuta et al[7], Fujii and Nakamoto[5], Izumino[9], Nakamoto and Sheth[11], Mustafaev and Shulman[10] and many others.

Keywords : Stationary distance vectors, Translatable radii.

Bjorck and Thomee[2] have shown that for a normal operator T,

$$M_T = sup_{\|f\|=1} \ \{\|Tf - (Tf,f)f\| = R_T,$$

where R_T is the radius of the smallest circle containing the spectrum. Garske[8] improved on the result to prove that for any bounded linear operator T,

$$M_T = sup_{\|f\|=1} \ \{\|Tf - (Tf,f)f\| \geq R_T.$$

Stampfli[15] proved that for a bounded linear operator T $\exists$ a unique complex scalar c_T, defined as the center of mass of T such that

$$\|T - c_T I\|^2 + |\lambda|^2 \leq \|T - c_T I + \lambda I\|^2, \ \ \forall \ \lambda \ \in C.$$

With the help of Stampfli's result Prasanna[14] proved that $M_T = \|T - c_T I\|$. Later Fujii and Prasanna[6] improved on the inequality of Garske to show that $M_T \geq w_T$ where w_T is the radius of the smallest circle containing the numerical range.
In [12] we proved that for any two bounded linear operators T and A if $0 \notin \sigma_{app} A$ then there exists a unique complex scalar λ_0 such that $\|T - \lambda_0 A\| \leq \|T - \lambda A\| \ \forall \lambda \ \in$ C. We defined $T - \lambda_0 A$ as the **minimal-norm translation of T in the direction of A** and proved that $\|T - \lambda_0 A\| = M_T(A)$. The equality of $\inf_\lambda \|T - \lambda A\| = M_T(A)$ was also studied by E.Asplund and V.Pt*á*k[1]
Then in [13] we introduced a couple of **translatable radii of an operator T in the direction of another operator A** as follows :
If 0 does not belong to the approximate point spectrum of A let

$$M_T(A) = sup_{\|f\|=1} \ \{\|Tf - \frac{(Tf,Af)}{(Af,Af)} Af\|\}$$

$$i.e., M_T(A) = sup_{\|f\|=1} \ \{\|Tf\|^2 - \frac{|\ (Tf,Af)\ |^2}{(Af,Af)}\}^{1/2}$$

and if $0 \notin \overline{W(A)}$, where $\overline{W(A)}$ stands for the closure of the numerical range of A, let

$$\tilde{M}_T(A) = sup_{\|f\|=1} \ \{\|Tf - \frac{(Tf,f)}{(Af,f)} Af\|\} \ .$$

We defined $M_T(A)$ and $\tilde{M}_T(A)$ as translatable radii of the operator T in the direction of A and proved in [13] that if $0 \notin \overline{W(A)}$ then

$$\tilde{M}_T(A) \geq M_T(A) \geq m_T(A)/\|A^{-1}\| \ ,$$

where $m_T(A)$ is the radius of the smallest circle containing the set $W_T(A) = \{ (Tf, Af)/(Af, Af) \ : \|f\| = 1 \}$.

Das[4] introduced the concept of stationary distance vectors while studying the eigenvalue problem Tf = λ f. Following the ideas of Das we here use the concept of stationary distance vectors to study the generalized eigenvalue problem Tf = λ Af and the translatable radius $M_T(A)$. We investigate the structure of the vectors for which the translatable radius $M_T(A)$ is attained and prove that if $M_T(A)$is attained at a vector f then $M_{T^*}(A^*)$ is attained at the vector $h/\|h\|$, where $h = Tf - (Tf, Af)/(Af, Af) \ Af$. We also show that if g is a state (normalized positive functional) on the Banach algebra B(H,H) of all bounded linear operators on H then

$$M_T(A) = sup\{ \ g(T^*T) - \frac{|\ g(A^*T)\ |^2}{g(A^*A)} \ \ : g \ \ is \ a \ state \ and \ \ g(A^*A) \neq 0\} \ .$$

The last result mentioned here is a generalization of a theorem of Williams [16].

2 Stationary distance vectors of the generalized eigenvalue problem Tf = λ Af

In this section we study the following :

" For any two bounded linear operators T and A what are the vectors that are nearest to or farthest from being eigenvectors of the equation $Tf = \lambda Af$ in the sense that $\|Tf - (Tf, Af)/(Af, Af) \ Af\|$ with unit f is minimum or maximum ?"

We give a necessary and sufficient condition that a unit vector f is at a stationary distance from being an eigenvector. We call such f's the stationary distance vectors and the corresponding λ = (Tf,Af)/(Af,Af) the stationary distance value of the eigenvalue problem $Tf = \lambda Af$. We use the concept of stationary vectors the definition of which is given below:

Definition 1 Stationary vector.
Let φ be a functional defined on the unit sphere of H. Then a unit vector f is said to be a stationary vector and φ is said to have a stationary value at f of φ iff the function $w_g(t)$ of a real variable t, defined as

$$w_g(t) = \varphi(\frac{f + tg}{\|f + tg\|})$$

has a stationary value at t=0 i.e., $w'_g(0) = 0$ for any arbitrary but fixed vector $g \in H$.
e.g., If $\varphi(f) = \|Tf - (Tf, Af)/(Af, Af) \ Af\|^2$ then a stationary vector f of functional φ is called the stationary distance vector of the eigenvalue problem $Tf = \lambda Af$.

We assume that 0 does not belong to the approximate point spectrum of A and prove the following theorem :

Theorem 1. The necessary and sufficient condition for a unit vector f to be a stationary distance vector of the generalized eigenvalue problem $Tf = \lambda Af$ is that it satisfies the following

$$(T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f$$

where $h = Tf - \lambda Af$ and $\lambda = \frac{(Tf,Af)}{(Af,Af)}$.

Proof. Consider $M_T(f) = \|Tf - (Tf,Af)/(Af,Af)\ Af\|$. Define the function $w_g(t)$ of a real variable t as follows

$$w_g(t) \ = \ {M_T}^2\ (\frac{f+tg}{\|f+tg\|}) \ = \ \frac{\|T(f+tg)\|^2}{\|f+tg\|^2} \ - \ \frac{|\ (T(f+tg), A(f+tg))\ |^{\ 2}}{(A(f+tg), A(f+tg))\ \|f+tg\|^2}$$

where g is arbitrary but fixed vector in H.

At a stationary vector f we have $w_g'(0) = 0$ and so

$$\begin{aligned}
&2\ Re\ (T^*Tf, g) - \|Tf\|^2\ 2\ Re(f,g) \ - \ \frac{\|Af\|^2}{\|Af\|^4}\ \ [\ (Tf, Af) \\
&\{\ \overline{(Tf,Ag) + (Tg,Af)}\ \}\ + \overline{(Tf,Af)}\{\ (Tf,Ag)\ +\ (Tg,Af)\ \}\ \ ] \\
&+ \frac{|\ (Tf,Af)\ |^2}{\|Af\|^4}\ \ \{\ \|Af\|^2\ 2\ Re\ (f,g)\ +\ 2\ Re\ (A^*Af, g)\ \} = 0\ .
\end{aligned}$$

Since g is arbitrary we get,

$$\begin{aligned}
&T^*Tf \ - \ \|Tf\|^2 f \ - \ \lambda\ T^*Af \ - \ \bar{\lambda}\ A^*Tf \ + \ \|Af\|^2\ \lambda^2 f + \lambda^2\ A^*Af = 0\ , \\
&\qquad \quad where \quad \lambda \ = \ (Tf,Af)/(Af,Af)\ .
\end{aligned}$$

Let $h = Tf - \lambda Af$, then $(h, Af) = 0$ and $\|h\|^2 = \|Tf\|^2 - |\ (Tf,Af)\ |^2/(Af,Af)$.
So we get

$$(T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f\ .$$

Thus the theorem is proved.

We now prove the following corollary :

Corollary 1. If $M_T(A)$ is attained at f then $M_{T^*}(A^*)$ is also attained at $h/\|h\| \ \ where \ \ h = Tf - (Tf,Af)/(Af,Af)\ Af$.

Proof. Suppose $M_T(A)$ is attained at a vector f and $\lambda = \frac{(Tf,Af)}{(Af,Af)}$. Then f is a stationary distance vector and so we get

$$\begin{aligned}
& (T^* - \bar{\lambda}A^*)(T - \lambda A)f = \|h\|^2 f \\
\Rightarrow \quad & (T^* - \bar{\lambda}A^*)h = \|h\|^2 f \\
\Rightarrow \quad & (T^*h, A^*h) = \bar{\lambda}(A^*h, A^*h) \\
\Rightarrow \quad & \bar{\lambda} = \frac{(T^*h, A^*h)}{(A^*h, A^*h)}
\end{aligned}$$

$$\begin{aligned}
Now \;\; T^*h &= \bar{\lambda}A^*h + \|h\|^2 f \\
\Rightarrow \|T^*h\|^2 &= \mid \bar{\lambda} \mid^2 \|A^*h\|^2 + \|h\|^4 \\
\Rightarrow \|T^*h\|^2 &= \|h\|^2 \{\|Tf\|^2 - \frac{\mid (Tf, Af) \mid^2}{(Af, Af)}\} + \frac{\mid (Tf, Af) \mid^2}{(Af, Af)} \cdot \frac{\|A^*h\|^2}{\|Af\|^2}
\end{aligned}$$

If the minimal-norm translation of T in the direction of A is T itself then the minimal-norm translation of T^* in the direction of A^* is also T^*. So if $M_T(A) = \|T\|$ then $M_{T^*}(A^*) = \|T^*\|$.
Let $M_T(A) = \|T\| = \|Tf\|, \;\; (Tf, Af)/(Af, Af) = 0$.
Then $M_{T^*}(A^*) = \|T^*\| = \|T\| = \|T^*h\|/\|h\|, \;\; since \;\; (Tf, Af)/(Af, Af) = 0.$
This completes the proof.

Next we prove the following theorem :

Theorem 2. Suppose T and A are two selfadjoint operators and f be a unit stationary distance vector such that (Tf,Af) is real, then f can be expressed as the linear combination of two eigenvectors of the problem $Tf = \lambda Af$.
Proof. As both T and A are selfadjoint and f is a stationary distance vector with (Tf,Af) real we get from the last theorem

$$(T - \lambda A)^2 f = \|h\|^2 f.$$

So we get

$$\begin{aligned}
\Rightarrow (T - \lambda A)^2 f \pm \|h\|h &= \|h\|^2 f \pm \|h\|h \\
\Rightarrow T(Tf - \lambda Af \pm \|h\|f) &= (\lambda A \pm \|h\|)(Tf - \lambda Af \pm \|h\|f)
\end{aligned}$$

$$Let \;\; g_1 = Tf - \lambda Af + \|h\|f$$

$$and \;\; g_2 = Tf - \lambda Af - \|h\|f \; .$$

Then we get

$$Tg_1 = (\lambda A + \|h\|)g_1 \;\; and \;\; Tg_2 = (\lambda A - \|h\|)g_2$$

so that

$$(T-\lambda A)g_1 = \|h\|g_1 \quad and \quad (T-\lambda A)g_2 = -\|h\|g_2 \ .$$

Thus $f = (g_1 - g_2)/(2\|h\|)$ completes the proof.

3 On the attainment of $M_T(A)$

Suppose $\{f_n\}$ be a sequence of unit vectors such that

$$\|Tf_n\|^2 - \frac{|\ (Tf_n, Af_n)\ |^2}{(Af_n, Af_n)} \quad \longrightarrow \quad M_T(A)^2.$$

As the unit sphere in H is weakly compact without loss of generality we may assume that $\{f_n\}$ converges weakly to f i,e, $f_n \rightharpoonup f$.

We now prove the following theorem :

Theorem 3. Suppose $\{f_n\}$ be a weakly convergent sequence of unit vectors such that

$$\|Tf_n\|^2 - \frac{|\ (Tf_n, Af_n)\ |^2}{(Af_n, Af_n)} \quad \longrightarrow \quad M_T(A)^2.$$

If the weak limit f is non-zero then $M_T(A)$ is attained for the vector $f/\|f\|$. If the supremum is not attained then all such sequences must tend weakly to zero.

Proof. Since $M_T(A)$ is translation invariant in the direction of A so without any loss of generality we may assume that the minimal-norm translation of T in the direction of A is T itself i,e, $M_T(A) = \|T\|$.

So there exists a sequence $\{f_n\},\ f_n \in H,\ \|f_n\| = 1$ such that $\|Tf_n\| \longrightarrow \|T\|$ and $(Tf_n, Af_n) \longrightarrow 0$. Considering the positive operator $\|T\|^2 I - T^*T$ we have

$$\begin{aligned}
& (\|T\|^2 f_n - T^*Tf_n, f_n) \longrightarrow 0 \\
\Rightarrow \quad & \|T\|^2 f_n - T^*Tf_n \longrightarrow 0\ , \quad \textit{by property of positive operators.} \\
If\ f \ \neq \quad & 0 \ we\ have \\
& \|T\|^2 (f_n, f) - (T^*Tf_n, f) \longrightarrow 0\ .
\end{aligned}$$

Since $f_n \rightharpoonup f$ and weak limit f is unique we get

$$\|T\|^2 = \frac{\|Tf\|^2}{\|f\|^2}\ .$$

The result that " if $f_n \rightharpoonup f,\ \|Tf_n\| \to \|T\|$ and $f \neq 0$ then $\|T\|$ is attained at $f/\|f\|$ " follows directly from the corollary 1 of Das[3].

As $M_T(T) = \|A\|$ the theorem is proved.

4 On generalization of a Theorem of Williams

Let $\mathcal{B}$ denote the set of all normalized positive linear functionals (states) on B(H,H) i.e.,

$$\mathcal{B} = \{ \ g \ : \ g \in L(B(H,H),C) \ \ and \ \ g(I) = 1 = \|g\| \ \}$$

Clearly $\mathcal{B}$ is $weak^*$compact. Let $\mathcal{P} = \{ \ g \ : \ g \in \mathcal{B} \ and \ g(A^*A) \neq 0 \ \}$.
Williams[16] proved that for any bounded linear operator T, $\|T\| \leq \|T - \lambda I\| \ \ \forall \lambda \ \in$ C iff there exists a state f such that $f(T^*T) = \|T^*T\|$ and f(T) = 0. We here show that if for two bounded linear operators T and A, $\|T\| \leq \|T - \lambda A\| \ \ \forall \ \lambda \ \in$ C then $\|T\|^2 \ = sup\{ \ \ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} \ \ : g \ \ is \ a \ state \ and \ \ g(A^*A) \neq 0\}$.
We now prove the following theorem :

Theorem 4. $[M_T(A)]^2 = sup\{ \ \ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} \ \ : g \ \ is \ a \ state \ and \ \ g(A^*A) \neq 0\}$.
Proof. Let $[S_T(A)]^2 = sup\{ \ \ g(T^*T) - \frac{|g(A^*T)|^2}{g(A^*A)} \ \ : g \ \ is \ \ a \ \ state \ \ and \ \ g(A^*A) \neq 0\}$.
Clearly $S_{T+\lambda A}(A) = S_T(A)$ and $M_{T+\lambda A}(A) = M_T(A)$ so that both are translation invariant in the direction of A. Without loss of generality we assume that $M_T(A) = \|T\|$.
Now for each $x \in H$, $\|x\| = 1$, let $g_x : B(H,H) \longrightarrow C$ be defined as $g_x(U) = (Ux, x) \ \ \forall U \ \in \ B(H,H)$.
Then g_x is a state and $g_x(A^*A) \neq 0$.
So

$$\begin{aligned}
\|T\| &= sup_{g_x}\{g_x(T^*T) - \frac{\mid g_x(A^*T) \mid^2}{g_x(A^*A)}\}^{1/2} \\
&\leq sup_{g \in \mathcal{P}}\{g(T^*T) - \frac{\mid g(A^*T) \mid^2}{g(A^*A)}\}^{1/2} \\
&\leq sup_{g \in \mathcal{P}}\{g(T^*T)\}^{1/2} \\
&= \|T\| \ .
\end{aligned}$$

This completes the proof.

Note. For A=I the result of Williams follows easily from Theorem 4.

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